Generic Properties of Column-Structured Matrices*

Takeo Yamada[†]
Department of Social Sciences
National Defense Academy
Yokosuka, 239 Japan

and

David G. Luenberger
Department of Engineering — Economic Systems
Stanford University
Stanford, California 94305

Submitted by Richard W. Cottle

ABSTRACT

A matrix is said to be a structured matrix (SM) if its entries are either fixed zeros or mutually independent free parameters. Several generic properties concerning the eigenvalues and eigenvectors of such a matrix are known. Very little is known, however, about the case where the entries of matrices are interdependent. If, for example, A, B, C, and D are mutually independent SMs (of dimension $n \times n$), then AB, A^{-1}, and $ABC + CD$ are not SMs in general, since the entries of these matrices are usually interrelated. However, the entries in these matrices are columnwise independent; that is, all elements in an arbitrary column can be simultaneously multiplied by an arbitrary scalar, without affecting other columns, and the result is still a matrix of the same class. A matrix with this property is referred to as a column-structured matrix (CSM). We investigate generic properties of CSMs, and establish nonrepeatedness, nonzeroness, and controllability of the nonzero eigenvalues and eigenvectors of such a matrix. These results have an important application to the problem of generic controllability of structured descriptor systems.

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[†]Formerly with the Department of Engineering-Economic Systems, Stanford University, Stanford, CA 94305.

52 Vanderbilt Ave., New York, NY 10017 0024-3795/85/$3.30

1. INTRODUCTION AND DEFINITIONS

A matrix is said to be a structured matrix (SM) if its entries are either fixed at zero or parameters which are mutually independent [1]. This class of matrices has been studied for a long time by many mathematicians. Frobenius, for example, proved that the determinant of such a matrix is an irreducible polynomial of its parameters, unless the matrix is itself reducible [2]. More recent works on this subject may be found in Ryser [3, 4], where SMs are defined with respect to an arbitrary field.

Many researchers in the area of control and system sciences have studied SMs, following the pioneering work of Lin [1] on structured controllability. Readers are referred to [5], [6], and [7] for more applications of SMs to the problems of control systems.

In this paper, we study the properties of SMs whose entries are not necessarily independent but are interdependent in some way. Before introducing this class of matrices, however, we begin with a very general concept.

DEFINITION 1 (Generalized structured matrix). A matrix $M = M(\xi_M)$ is a *generalized structured matrix* (GSM) if each entry of M is a member of $F(\xi_M)$, where $F(\xi_M)$ is the field of all rational functions with real coefficients in indeterminates $\xi_M = (\xi_1, \xi_2, \dots, \xi_{\rho(M)})$.

Here, $\rho(M)$ denotes the number of free parameters defining M, and the domain of ξ_M is $R^{\rho(M)}$, which is also referred to as the parameter space. The matrix M is also denoted $M(\xi_M)$, or more simply $M(\xi)$, to indicate the parameters explicitly.

Given a GSM $M(\xi)$, a mapping $\Pi: R^{\rho(M)} \to \{0,1\}$ is said to be a *property* of M, where $\Pi(\xi)=1$ (0) means that Π holds (fails) at $\xi \in R^{\rho(M)}$. Π is *generic* if $\operatorname{Ker}\Pi \subseteq V$ holds for some proper algebraic variety [5] V. Since the Lebesgue measure of a proper algebraic variety is zero, this implies that a generic property is a property that holds almost everywhere in the parameter space. For SMs, several properties are known to be generic; irreducibility is an example of this kind, and the rank of a SM is also generic. See [6] and [7] for these topics.

We wish to investigate generic properties of GSMs. However, perhaps this concept is too general to lead to meaningful results. The following defines a class of matrices which lies between SMs and GSMs.

DEFINITION 2 (Column-structured matrix). A GSM $M = M(\xi)$ of dimension $m \times n$ is said to be a *column-structured matrix* (CSM) if there exist n

rational mappings

$$\eta_i(\cdot,\cdot): R^{\rho(M)}\times R \to R^{\rho(M)}, \qquad i=1,2,\dots,n,$$

with η_i associated with the ith column of M, such that for every $i=1,2,\dots,n$,

$$M(\eta_i(\xi,\alpha_i)) = M(\xi)\operatorname{diag}(1,\dots,1,\alpha_i,1,\dots,1)$$

and

$$\eta_i(\xi,1)=\xi.$$

One of the motivations for this definition is the structured descriptor system (or generalized state-space system) [8–11] of the following form:

$$Ex(t+1) = Ax(t)+Bu(t), \qquad t=0,1,\dots,$$

where $x(\cdot)\in R^n$, $u(\cdot)\in R^m$, and E, A, B are SMs of respective dimensions $n\times n$, $n\times n$, and $n\times m$. If E is generically nonsingular, this becomes

$$x(t+1) = \bar{A}x(t)+\bar{B}u(t), \qquad t=0,1,\dots,$$

where the system matrix $(\bar{A},\bar{B}) = E^{-1}(A,B)$ is clearly a CSM. More complicated examples of CSMs are given below, but complete characterization of the class of CSMs seems difficult.

Examples. *Let A, B, C and D be mutually independent $n\times n$ SMs, and assume that A is generically nonsingular. Then, the matrices AB, A^{-1}, and $ABC+CD$ are all* CSMs.

Proof. This is obvious for AB and A^{-1}. To prove the third case, let us denote the free parameters included in (A,B,C,D) by ξ, and let $M(\xi) = ABC+CD$. For an arbitrary scalar α, define

$$A_\alpha \triangleq A, \qquad B_\alpha \triangleq B,$$

$$C_\alpha \triangleq C\operatorname{diag}(1,\dots,1,\alpha,1,\dots,1),$$

$$D_\alpha \triangleq \operatorname{diag}(1,\dots,1,\alpha^{-1},1,\dots,1)\,D\operatorname{diag}(1,\dots,1,\alpha,1,\dots,1).$$

The transformation of the matrices from (A,B,C,D) to $(A_\alpha, B_\alpha, C_\alpha, D_\alpha)$

naturally induces the transformation of its entries [from ξ to, say, $\eta_i(\xi,\alpha)$]. Then we have

$$M(\eta_i(\xi,\alpha)) = A_\alpha B_\alpha C_\alpha + C_\alpha D_\alpha = (ABC + CD)\operatorname{diag}(1,\dots,1,\alpha,1,\dots,1),$$

which completes the proof. ∎

Definition 2 says that the columns of a CSM can be freely multiplied by arbitrary scale factors. More precisely, we introduce column scalings for a CSM $M = M(\xi)$ in the following way. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in R^n$ be a vector of scale factors. Define the parameters η by applying the transformations $\eta_1, \eta_2, \dots, \eta_n$ successively to ξ; i.e.,

$$\eta = \eta_n(\cdots \eta_2(\eta_1(\xi,\alpha_1),\alpha_2),\dots,\alpha_n).$$

For simplicity, we denote this as $\eta = \eta(\xi,\alpha)$. The resulting matrix $M(\eta)$, also denoted hereafter as $M(\alpha)$, is given by $M(\alpha) = M\operatorname{diag}(\alpha_1,\alpha_2,\dots,\alpha_n)$.

The purpose of this paper is to investigate the generic properties of CSMs. More specifically, for an $n \times n$ CSM $A(\xi_A)$, we shall explore the properties of some elements of the field $F(\xi_A)$ (and its extensions), which are closely related to the spectral properties of $A(\xi_A)$. The results are also interpreted in terms of genericity.

2. IRREDUCIBILITY OF COLUMN-STRUCTURED MATRICES

Let A be an $n\times n$ GSM. A is said to be reducible if for some permutation matrix P and square GSMs A_{11} and A_{22},

$$P'AP = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}.$$

Then, the characteristic polynomial $\phi_A(\lambda) = \det(A - \lambda I)$ can be decomposed as

$$\phi_A(\lambda) = \phi_{A_{11}}(\lambda)\phi_{A_{22}}(\lambda),$$

where $\phi_{A_{ii}}(\lambda)$ is the characterisitc polynomial of A_{ii} $(i = 1,2)$. This section discusses the converse of this fact.

It is convenient to introduce the graph representation $G(A)$ [12] to investigate the irreducibility of $A = (a_{ij})$. $G(A)$ is the graph with nodes

$N=\{1,2,\ldots,n\}$ and the set of arcs $W \subseteq N \times N$, where $(i,j)\in W$ if and only if $a_{ji} \neq \bar{0}$ $(i,j=1,2,\ldots,n)$.[1] Then a necessary condition for A to be irreducible is given by the following lemma.

LEMMA 1. *If A is an irreducible* GSM, *then $G(A)$ is strongly connected.*

For the proof and graph-theoretic terminology, see [12] or [13].

For a GSM $A=A(\xi_A)$, let $F(\xi_A)[\lambda]$ denote the ring of all $F(\xi_A)$-coefficient polynomials of λ. Corresponding to an $n\times n$ GSM A, its generalized trace of order k is defined by

$$\operatorname{tr}_k(A)=\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n}\det(A[i_1,i_2,\ldots,i_k]),$$

where $A[i_1,i_2,\ldots,i_k]$ is the submatrix of A obtained from the columns and rows $i_1,i_2,\ldots,i_k$, and the summation ranges over all possible combinations of integers satisfying the inequality condition. Two familiar examples are:

$$\operatorname{tr}_1(A)=\operatorname{tr}(A)\qquad \text{(trace in the usual sense)},$$

$$\operatorname{tr}_n(A)=\det(A).$$

DEFINITION 3 (Generic order). Let A be an $n\times n$ GSM. Its *generic order* is defined as

$$\nu(A)=\max\{k\,|\operatorname{tr}_k(A)\neq\bar{0}\}.$$

Since $\phi_A(\lambda)=\det(A-\lambda I)=(-\lambda)^n+(-\lambda)^{n-1}\operatorname{tr}_1(A)+\cdots+\operatorname{tr}_n(A)$, and $\operatorname{tr}_k(A)=0$ for $k=\nu(A)+1,\ldots,n$, we obtain

$$\phi_A(\lambda)=(-\lambda)^{n-\nu(A)}p_A(\lambda),$$

where

$$p_A(\lambda)=(-\lambda)^{\nu(A)}+\operatorname{tr}_1(A)(-\lambda)^{\nu(A)-1}+\cdots+\operatorname{tr}_{\nu(A)}(A).$$

The polynomial $p_A(\lambda)$ may be appropriately called the nonzero part of the

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[1] $\bar{0}$ denotes the zero element in $F(\xi_A)$ or its extensions.

characteristic polynomial $\phi_A(\lambda)$, since A has $\nu(A)$ nonzero eigenvalues and an $n-\nu(A)$ times repeated eigenvalue which is 0.

It should be clear that the generic order of a GSM is actually a generic number, since by definition of $\nu(A)$, $V=\{\xi\in R^{\rho(A)}|\mathrm{tr}_{\nu(A)}(A)=0,\ A=A(\xi)\}$ is indeed a proper algebraic variety. To obtain further results, however, we need to restrict ourselves to the class of CSMs. The first result is the following:

THEOREM 1 (Reducibility theorem). *Let* $A(\xi_A)$ *be an* $n\times n$ CSM. *If there exist two polynomials* $p_1(\lambda)$ *and* $p_2(\lambda)$ *in* $F(\xi_A)[\lambda]$ *such that*

$$p_A(\lambda)=p_1(\lambda)p_2(\lambda),$$

$$0<\deg(p_i)<\nu(A)\qquad i=1,2,$$

then A *is reducible.*

Proof. Let $A=A(\xi)$, $\xi=(\xi_1,\xi_2,\dots,\xi_{\rho(A)})$, and suppose that

$$p_A(\lambda)=p_1(\lambda)p_2(\lambda),\tag{1}$$

where $p_i(\lambda)\in F(\xi)[\lambda]$ $(i=1,2)$. Without loss of generality, it can be assumed that no canceling factors occur in $p_1(\lambda)$ and $p_2(\lambda)$. Substituting $\eta=\eta(\xi,\alpha)$ in place of ξ in (1) yields

$$p_{A(\alpha)}(\lambda)=p_1(\lambda,\alpha)p_2(\lambda,\alpha)\tag{2}$$

where

$$p_i(\lambda,\alpha)\in F(\xi,\alpha)[\lambda]\qquad(i=1,2).$$

Since $A(1)=A$ and $A(0)=0$, we have

$$p_i(\lambda,1)=p_i(\lambda),\quad p_i(\lambda,0)=\lambda^{r_i}\qquad(i=1,2),\tag{3}$$

where $r_i=\deg(p_i)$.

Furthermore, there can be no canceling factors in $p_1(\lambda,\alpha)$ and $p_2(\lambda,\alpha)$, since otherwise $p_1(\lambda)$ and $p_2(\lambda)$ would have a canceling factor by setting $\alpha=I$. Then, since $p_{A(\alpha)}(\lambda)$ is an affine function of α_i $(i=1,2,\dots,n)$, at most one of $p_1(\lambda,\alpha)$ and $p_2(\lambda,\alpha)$ can be a nonconstant affine function of α_i. Let us say that column i *affects* $p_k(\lambda)$ $(k=1,2)$ if $p_k(\lambda,\alpha)$ is a nonconstant affine function of α_i. Let us partition the set of nodes in $G(A)$ as

$$N_1 = \{\, i \mid \text{column } i \text{ affects } p_1(\lambda)\},$$

$$N_2 = \{\, i \mid \text{column } i \text{ does not affect } p_1(\lambda)\}.$$

Note that columns in N_1 do not affect $p_2(\lambda)$, and any column which affects $p_2(\lambda)$ necessarily belongs to N_2. If N_1 is empty, no columns affect $p_1(\lambda)$. In that case, $p_1(\lambda)=p_1(\lambda,1)=p_1(\lambda,0)=\lambda^{r_1}$, since $p_1(\lambda,\alpha)$ is independent of α. This is a contradiction, since $p_A(0)=p_1(0)p_2(0)\neq 0$. Similarly, if N_2 is empty, no columns affect $p_2(\lambda)$, leading to a similar contradiction. Thus, N_1 and N_2 give a nontrivial disjoint partition of N.

Now assume that A is irreducible. Then, by Lemma 1, $G(A)$ is strongly connected. Therefore, for any pair of nodes $i\in N_1$ and $j\in N_2$, there exists a cycle in $G(A)$ containing these two nodes. Take a cycle of minimum length which contains nodes both from N_1 and N_2. Such a cycle is illustrated in Figure 1. By an appropriate renumbering of nodes, we can assume that this cycle is indexed as $(1,2,\dots,r,r+1,\dots,t=r+s)$, where $\{1,2,\dots,r\}\in N_1$ and $\{r+1,\dots,t\}\in N_2$. Note that there can be no arcs (i,j) for $j>i+2$ or for $i\in N_2$ and $j\in N_1$ in $G(A)$, since in this case we would have a shorter cycle. Thus, A has the form of the following matrix, where elements on this cycle are all nonzero:

$$A=\left[\begin{array}{ccccc|cccc|c}
* & * & * & \cdots & * & 0 & \cdots & 0 & \beta_t & \\
\beta_1 & * & * & \cdots & * & & & & 0 & \\
 & \beta_2 & * & & \vdots & & \mathbf{0} & & \vdots & * \\
 & & \ddots & \ddots & & & & & & \\
\mathbf{0} & & & \beta_{r-1} & * & & & & 0 & \\
\hline
0 & 0 & \cdots & 0 & \beta_r & * & * & \cdots & * & \\
 & & & & 0 & \beta_{r+1} & * & \cdots & * & \\
 & & \mathbf{0} & & \vdots & & \ddots & \ddots & \vdots & * \\
 & & & & 0 & \mathbf{0} & & \beta_{t-1} & * & \\
\hline
 & & * & & & & * & & & *
\end{array}\right]. \tag{4}$$

Let A_1 and A_2 denote the first and second matrices on the diagonal block of (4) respectively.

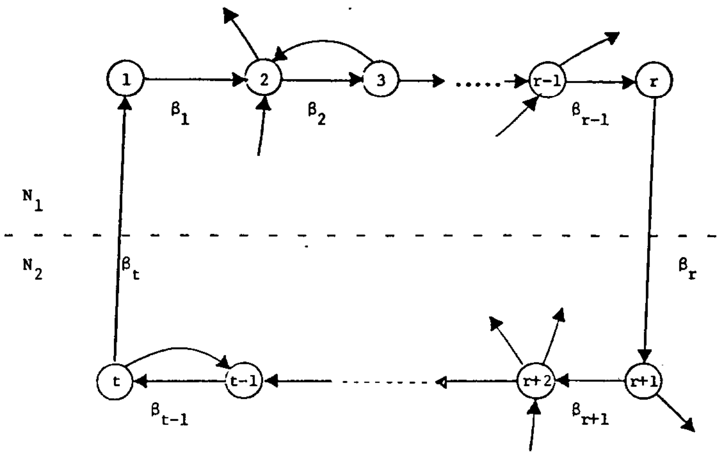

FIG. 1. Minimum-length cycle ranging over N_1 and N_2.

Now, consider the scaled matrix $A(\alpha)$, and let $\alpha \to \tilde{1}_t$, where $\tilde{1}_t$ is the n-dimensional vector of the form $(1,1,\ldots,1,0,0,\ldots,0)$ with its first t elements being 1. Then we have

$$\phi_{A(\alpha)}(\lambda) \to (-\lambda)^{n-t}\det(A[1,2,\ldots,t]-\lambda I)$$

$$=(-\lambda)^{n-t}\det\left[\begin{array}{c|c} A_1-\lambda I & \begin{matrix} & \beta_t \\ 0 & \end{matrix} \\ \hline \begin{matrix} & \beta_r \\ 0 & \end{matrix} & A_2-\lambda I \end{array}\right].$$

The last determinant is clearly affine in β_t; say $c_1 + c_2\beta_t$. Setting $\beta_t = 0$ shows that $c_1 = \det(A_1 - \lambda I)\det(A_2 - \lambda I)$. The value of c_2 is the $(1,t)$ cofactor, which is $(-1)^{1+t}\beta_1\beta_2\cdots\beta_t$. Thus, as we let $\alpha \to \tilde{1}_t$,

$$\phi_{A(\alpha)}(\lambda) \to (-\lambda)^{n-t}\left\{\det(A_1-\lambda I)\det(A_2-\lambda I)+(-1)^{1+t}\beta_1\beta_2\cdots\beta_t\right\}. \tag{5}$$

Similarly, $\phi_{A(\alpha)}(\lambda) = (-\lambda)^{n-\nu(A)}p_1(\lambda,\alpha)p_2(\lambda,\alpha)$ becomes

$$\phi_{A(\alpha)}(\lambda) \to (-\lambda)^{n-t'}\bar{p}_1(\lambda)\bar{p}_2(\lambda) \qquad (\text{as} \quad \alpha \to \tilde{1}_t), \tag{6}$$

where $\bar{p}_1(\lambda)$ and $\bar{p}_2(\lambda)$ are polynomials in $F(\xi_A)[\lambda]$ with respective orders r' and s', and $t' = r' + s'$.

Next, let $\alpha \to \tilde{1}_r$. Then, since $A_2 \to 0$ and $\beta_t \to 0$,

$$\phi_{A(\alpha)}(\lambda) \to (-\lambda)^{n-r} \det(A_1 - \lambda I). \tag{7}$$

Since all columns in N_2 go to zero as $\alpha \to \tilde{1}_r$, (3) implies

$$\bar{p}_2(\lambda) \to (-\lambda)^{s'} \qquad (\text{as} \quad \alpha \to \tilde{1}_r).$$

Furthermore, $\bar{p}_1(\lambda)$ is not affected by this limit operation. Thus, we obtain

$$(-\lambda)^{n-r} \det(A_1 - \lambda I) = (-\lambda)^{n-r'} \bar{p}_1(\lambda).$$

Similarly,

$$(-\lambda)^{n-s} \det(A_2 - \lambda I) = (-\lambda)^{n-s'} \bar{p}_2(\lambda).$$

Substituting these into (5) and using (6) yields

$$\begin{aligned}(-\lambda)^{n-r'} \bar{p}_1(\lambda)(-\lambda)^{n-s'} \bar{p}_2(\lambda) + (-\lambda)^{2n-t} \beta_1 \beta_2 \cdots \beta_t (-1)^{1+t} \\ = (-\lambda)^{n-r'} \bar{p}_1(\lambda)(-\lambda)^{n-s'} \bar{p}_2(\lambda).\end{aligned}$$

Therefore, $\beta_1 \beta_2 \cdots \beta_t = \bar{0}$, which is a contradiction. ∎

This theorem provides a very powerful tool for analyzing the properties of CSM's. One application is the following theorem.

THEOREM 2 (Common-root theorem). *Let $A(\xi_A)$ be a CSM, and let $w(\lambda)$ be a polynomial in $F(\xi_A)[\lambda]$ with $1 \leqslant \deg(w) < \nu(A)$. If $\phi_A(\lambda)$. If $\phi_A(\lambda)$ and $w(\lambda)$ have a nonzero root in common, then A is reducible.*

Proof. We can assume, without loss of generality, that $w(\lambda)$ does not contain the factor λ, because if $w(\lambda) = \lambda^k \bar{w}(\lambda)$ for some $k > 0$, we can use $\bar{w}(\lambda)$ instead of $w(\lambda)$.

Let $\phi_A(\lambda) = Q_1(\lambda) w(\lambda) + w_1(\lambda)$, where Q_1 and w_1 are both in $F(\xi_A)[\lambda]$, and $\deg(w_1) < \deg(w)$. If $w_1(\lambda) = 0$,

$$p_A(\lambda) = \left\{(-\lambda)^{-n+\nu(A)} Q_1(\lambda)\right\} w(\lambda)$$

is a proper decomposition of $p_A(\lambda)$. Then, by Theorem 1, A is reducible.

If $w_1(\lambda) \neq 0$, $\phi_A(\lambda)$ and $w_1(\lambda)$ have a nonzero root in common. Then we can repeat the above argument using w_1 instead of w. Continuing in this way, we either have $w_k = 0$ for some $k > 0$, or $w_k =$ nonzero constant. If the former case occurs, we again have a nontrivial decomposition of $p_A(\lambda)$, implying that A is reducible. The latter can never happen, since w_k must have a root in common with $\phi_A(\lambda)$. ■

3. EIGENVALUES AND EIGENVECTORS OF COLUMN-STRUCTURED MATRICES

Using the theorems of the previous section, we explore the structure of eigenvalues and eigenvectors of irreducible CSM's.

THEOREM 3 (Nonrepeatedness of nonzero eigenvalues). *Let A be an $n \times n$ irreducible* CSM. *Then A has $\nu(A)$ nonzero eigenvalues which are mutually distinct.*

Proof. If, on the contrary, A has a nonzero eigenvalue which is repeated, then $\phi_A(\lambda)$ and $p'_A(\lambda)$ have that root in common. Since $\deg(p'_A) < \nu(A)$, Theorem 2 then implies that A is reducible, which is a contradiction. ■

For reducible matrices, we have:

THEOREM 4. *Let A be an $n \times n$* CSM *of generic order $\nu(A)$. Then A has $\nu(A)$ nonzero eigenvalues which are mutually distinct.*

Proof. A can be put in the following form:

$$A = \begin{bmatrix} A_{11} & & & 0 \\ A_{21} & A_{22} & & \\ \vdots & \vdots & \ddots & \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix},$$

where A_{ii} $(i = 1, 2, \dots, k)$ are irreducible (or possibly some of the A_{ii} are zero matrices of order 1). By Theorem 3, there exist $\nu(A_{ii})$ nonzero eigenvalues of A_{ii} which are mutually distinct. Nonzero eigenvalues of A_{ii} and A_{jj} $(i \neq j)$ cannot be identical, since by multiplying the ith block column of A by an arbitrary scalar, we can arbitrarily change the eigenvalues associated with this block without changing the eigenvalues of other blocks. Since

$$\nu(A) = \sum_{i=1}^{k} \nu(A_{ii}),$$

the theorem is proved. ■

THEOREM 5 (Nonzeroness of eigenvectors). *Let A be an irreducible CSM. Then the right and left eigenvectors associated with the $\nu(A)$ nonzero eigenvalues have no fixed zero components.*

Proof. Let λ be a nonzero eigenvalue and x be the right eigenvector associated with λ. Assume x has some fixed zero components. Then, by an appropriate permutation of columns and rows, x can be written as

$$x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix},$$

where x_1 is a c-vector representing the nonzero elements of x, and x has $n - c > 0$ elements of fixed zeros. Partitioning A correspondingly, we have

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ 0 \end{bmatrix},$$

that is,

$$A_{11} x_1 = \lambda x_1 .$$

Therefore, λ is also a solution to $p_{A_{11}}(\lambda) = 0$. Since $\nu(A_{11}) \leqslant \nu(A)$, there are two possibilities:

(i) $\nu(A_{11}) < \nu(A)$. In this case, $\phi_A(\lambda)$ and $p_{A_{11}}(\lambda)$ have a root in common, and $\deg(p_{A_{11}}) < \nu(A)$. Then, from Theorem 2, A is reducible, which is a contradiction.

(ii) $\nu(A_{11}) = \nu(A)$. In this case, λ is a solution both to $p_A(\lambda) = 0$ and $p_{A_{11}}(\lambda) = 0$ simultaneously. Furthermore, $\deg(p_{A_{11}}) = \deg(p_A)$, and the highest orders of these polynomials are both $(-\lambda)^{\nu(A)}$. Therefore, we obtain

$$p_A(\lambda) = p_{A_{11}}(\lambda) + q(\lambda), \qquad \deg(q) < \nu(A).$$

If $q \neq 0$, then $\phi_A(\lambda)$ and $q(\lambda)$ have a common root, which by Theorem 2 implies that A is reducible, contradicting the assumption of irreducibility.

The only remaining possibility is the case of $q(\lambda)=0$, or equivalently, $p_A(\lambda)=p_{A_{11}}(\lambda)$. However, this is impossible by the next lemma.

Similar arguments prove nonzeroness of the elements of left eigenvectors. ∎

LEMMA 2. *Let A be an $n\times n$ irreducible* CSM, *and let A_{11} be its $c\times c$ principal submatrix* $(0<c<n)$. *Then $p_A(\lambda)\neq p_{A_{11}}(\lambda)$.*

Proof. By appropriate permutation, A_{11} can be assumed to be the $c\times c$ submatrix in the top left corner of A. Assume that $\nu(A)=\nu(A_{11})$ and $p_A(\lambda)=p_{A_{11}}(\lambda)$. Let

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Consider the graph $G(A)$, and define the partition of its nodes by

$$N_1=\{1,2,\dots,c\},\qquad N_2=\{c+1,c+2,\dots,n\}.$$

Since A is irreducible, there exists a cycle for any pair $i\in N_1$ and $j\in N_2$ that contains these points. Take the minimum-length cycle containing points from both N_1 and N_2. By appropriate renumbering, we can write this cycle as

$$(1,2,\dots,r,r+1,\dots,t)$$

where

$$\{1,2,\dots,r\}\subseteq N_1,\qquad \{r+1,r+2,\dots,t\}\subseteq N_2.$$

Then, by minimality of cycle length, A must have the form of (4). (See Figure 1.)

Since

$$\phi_A(\lambda)=\det(A-\lambda I)=(-\lambda)^{n-\nu(A)}p_A(\lambda),$$

and

$$\phi_{A_{11}}(\lambda)=\det(A_{11}-\lambda I)=(-\lambda)^{c-\nu(A)}p_{A_{11}}(\lambda),$$

we have

$$\det(A-\lambda I)=(-\lambda)^{n-c}\det(A_{11}-\lambda I).$$

The change of parameter from ξ to $\eta=\eta(\xi,\alpha)$ implies

$$\det(A(\alpha)-\lambda I)=(-\lambda)^{n-c}\det(A_{11}(\alpha)-\lambda I). \tag{8}$$

Now, let $\alpha\to\tilde{1}_t$. This yields, as in (5),

$$\det(A(\alpha)-\lambda I)\to(-\lambda)^{n-t}\{\det(A_1-\lambda I)\det(A_2-\lambda I)$$
$$+(-1)^{t+1}\beta_1\beta_2\cdots\beta_t\},$$

where A_1 and A_2 are, respectively, the first and second submatrices on the diagonal block of (4). Furthermore, since A_1 is a submatrix of A_{11},

$$\det(A_{11}(\alpha)-\lambda I)\to(-\lambda)^{c-r}\det(A_1-\lambda I)\qquad(\text{as}\quad \alpha\to\tilde{1}_t).$$

Substituting these into (8) yields

$$\det(A_1-\lambda I)\det(A_2-\lambda I)+(-1)^{1+t}\beta_1\beta_2\cdots\beta_t=(-\lambda)^s\det(A_1-\lambda I) \tag{9}$$

Substituting a solution to $\det(A_1-\lambda I)=0$ into (9) yields

$$\beta_1\beta_2\cdots\beta_t=0,$$

which is a contradiction. ■

4. CONTROLLABILITY OF NONZERO EIGENVALUES

In this section, we consider the following single-input *column-structured system*:

$$x(t+1)=Ax(t)+bu(t),$$

where (A,b) is an $n\times(n+1)$ CSM with irreducible A and nonzero b. All modes associated with nonzero eigenvalues will be shown to be controllable [8]. Before showing this, however, we need a lemma.

Define the polynomials $g_i(\lambda)$, $i=1,2,\dots,n$, by

$$g_i(\lambda)=\{\mathrm{adj}(A-\lambda I)\}_i b,$$

where $\{\mathrm{adj}(A-\lambda I)\}_i$ is the ith row of the adjugate matrix [14] of $A-\lambda I$.

LEMMA 3. *Given an* $n\times(n+1)$ CSM (A,b) *with irreducible* A *and nonzero vector* b, *there exists an* i $(1\leqslant i\leqslant n)$ *such that*

$$p_A(\lambda)\nmid g_i(\lambda).$$

Proof. Let $r=\nu(A)$. Without loss of generality, by appropriate permutation if necessary, we can assume that $\mathrm{rank}(A_{rr})=r$, where A_{rr} is the $r\times r$ principal minor of A in the top left corner.

Assume, contrary to the conclusion of the lemma, that

$$p_A(\lambda)|g_i(\lambda)\qquad\text{for all}\quad i=1,2,\dots,n.$$

Since $\mathrm{adj}(A-\lambda I)(A-\lambda I)=\det(A-\lambda I)I$, we have

$$\{\mathrm{adj}(A-\lambda I)\}_i Ab=\det(A-\lambda I)\,b_i+\lambda g_i(\lambda).$$

Therefore,

$$p_A(\lambda)|\{\mathrm{adj}(A-\lambda I)\}_i Ab\qquad\text{for all } i.$$

Similarly,

$$p_A(\lambda)|\{\mathrm{adj}(A-\lambda I)\}_i A^j b\qquad\text{for all}\quad i \text{ and } j\geqslant 0. \tag{10}$$

Let k be the minimum number of j such that the first r elements of $A^j b$ are not all zero, i.e.,

$$A^j b=\left[\frac{0}{*}\right]\}\,r\qquad(j<k),$$

$$A^k b=\begin{bmatrix} b_1^{(k)}\\ \vdots\\ b_n^{(k)}\end{bmatrix},\qquad b_i^{(k)}\neq 0\quad\text{for some}\ \ 1\leqslant i\leqslant r.$$

If we consider the graph $G(A,b)$, which is obtained by adding to the graph

$G(A)$ a node $\{u\}$ and directed arcs $\{(u,i)\,|\,b_i \neq 0,\ i=1,2,\ldots,n\}$, then k is the minimum length of paths connecting u to some nodes in $\{1,2,\ldots,r\}$. Since $G(A)$ is strongly connected, k is finite.

Now, consider the scaled matrix $A(\alpha)$, with

$$\alpha = (\underbrace{1,1,\ldots,1}_{r}, \bar{\alpha}, \bar{\alpha}, \ldots, \bar{\alpha}),$$

and define

$$g_{i,\alpha}^{(k)}(\lambda) = \{\operatorname{adj}(A(\alpha)-\lambda I)\}_i A(\alpha)^k b.$$

By changing the parameter from ξ to $\eta = \eta(\xi, \alpha)$, (10) becomes

$$p_{A(\alpha)}(\lambda) | g_{i,\alpha}^{(k)}(\lambda). \tag{11}$$

Since $A(\alpha)$ has the form of

$$A(\alpha) = \left[\begin{array}{c|c} A_{rr} & \bar{\alpha}A_{r*} \\ \hline \underbrace{A_{*r}}_{r} & \bar{\alpha}A_{**} \end{array}\right] \begin{array}{l} \}\, r \\ \end{array},$$

$$\phi_{A(\alpha)}(\lambda) = \det(A(\alpha)-\lambda I) = (-\lambda)^{n-r}\det(A_{rr}-\lambda I) + O(\bar{\alpha}),$$

where $O(\bar{\alpha})$ denotes the term of order $\bar{\alpha}$ or higher. Since $\nu(A(\alpha)) = \nu(A)$, this implies

$$p_{A(\alpha)}(\lambda) = \det(A_{11}-\lambda I) + O(\bar{\alpha}). \tag{12}$$

Let us now examine the cofactors of $(A(\alpha)-\lambda I)$:

$$\begin{aligned} &j\text{th element of } \{\operatorname{adj}(A(\alpha)-\lambda I)\}_i \\ &\quad = (j,i) \text{ cofactor of } A(\alpha)-\lambda I \\ &\quad = \begin{cases} \det(A_{rr}^{(i)}-\lambda I)(-\lambda)^{n-r} + O(\bar{\alpha}) & (j=i), \\ d_{r-2}(\lambda)(-\lambda)^{n-r} + O(\bar{\alpha}) & (j \neq i,\quad 1 \leqslant j \leqslant r), \\ O(\bar{\alpha}) & (r+1 \leqslant j \leqslant n), \end{cases} \end{aligned}$$

where $A_{rr}^{(i)}$ is the $(r-1)\times(r-1)$ submatrix of A_{rr} with ith column and row deleted, and $d_{r-2}(\lambda)$ is a polynomial of degree at most $r-2$.

Since

$$A(\alpha)^k b = \begin{bmatrix} b_1^{(k)}\bar{\alpha}^k \\ \vdots \\ b_n^{(k)}\bar{\alpha}^k \end{bmatrix},$$

we have

$$g_{i,\alpha}^{(k)}(\lambda) = \left\{ b_i^{(k)} \det\left(A_{rr}^{(i)} - \lambda I\right) + d_{r-2}(\lambda) \right\} (-\lambda)^{n-r} \bar{\alpha}^k + O(\bar{\alpha}^{k+1}). \tag{13}$$

Substituting (12) and (13) into (11) yields

$$\det(A_{rr} - \lambda I) | b_i^{(k)} \det\left(A_{rr}^{(i)} - \lambda I\right) + d_{r-2}(\lambda).$$

This is a contradiction, since with $b_i^{(k)} \neq 0$ the right-hand side is a polynomial of order $r - 1$, which cannot be divided by a polynomial of order r. ■

We are now ready to state the theorem.

THEOREM 6 (Controllability of nonzero modes). *Suppose (A, b) is an $n \times (n+1)$ CSM with irreducible A and nonzero b. Then every mode of A associated with a nonzero eigenvalue is controllable. (This means that $y'b \neq \bar{0}$ holds for any left eigenvector y' associated with a nonzero eigenvalue of A.)*

Proof. Let ζ be a nonzero eigenvalue of A, and let y' be the left eigenvector associated with it. If it is not controllable, then be definition, $y'b = \bar{0}$. Since ζ is a nonrepeated eigenvalue of A (Theorem 3), y' is proportional to any one of nonzero rows of $\mathrm{adj}(A - \zeta I)$ (see [8]). Thus, we have

$$\{\mathrm{adj}(A - \zeta I)\}_i b = \bar{0} \qquad \text{for all} \quad i = 1, 2, \ldots, n. \tag{14}$$

Now, take i such that

$$P_A(\lambda) \nmid g_i(\lambda). \tag{15}$$

This is possible by Lemma 3. Then, from (14), ζ is a solution both to $\phi_A(\lambda) = 0$ and to $g_i(\lambda) = 0$. Define $w(\lambda)$ by

$$g_i(\lambda) = Q(\lambda) p_A(\lambda) + w(\lambda), \qquad \deg(w) < \deg(P_A) = \nu(A).$$

Then, by (15), $w(\lambda) \neq \bar{0}$. Theorem 2 then implies A is reducible. This is a contradiction. ■

5. REMARKS ON GENERICITY AND RELATED WORKS

Let $A = A(\xi_A)$ be an $n \times n$ CSM defined over $R^{\rho(A)}$. Denote the Sylvester resultant for two polynomials $f_i(\lambda) \in F(\xi_A)[\lambda]$ $(i = 1,2)$ as $\Delta(f_1, f_2)$. It is well known that $f_1(\lambda) = 0$ and $f_2(\lambda) = 0$ have a root in common if and only if $\Delta(f_1, f_2) = 0$. Then Theorem 4 implies that $V = \{\xi_A \in R^{\rho(A)} \mid \Delta(\phi_A, p'_A) = 0\}$ is a proper algebraic variety. Therefore, Theorem 4 can be restated as:

THEOREM 4′. *An $n \times n$ CSM A has $\nu(A)$ generically nonzero eigenvalues which are generically nonrepeated.*

Similarly, Theorems 5 and 6 imply generic nonzeroness of the components of the eigenvectors associated with generically nonzero eigenvalues of an irreducible CSM, and generic controllability of such eigenvalues.

Finally, some comments on related works are in order. Although the idea of a CSM and its properties are believed to be new, similar results have been known for the limited case of SMs. Theorem 1 is a generalization of the main theorem of Ryser [4]. The SM version of Theorem 3 was proved by Shields and Pearson [6], and also by Hosoe and Matsumoto [7]. Shields and Pearson also proved Theorem 5 for the case of SMs, with the additional assumption of nonsingularity of A. Furthermore, Hosoe and Matsumoto [7] proved the SM version of Theorem 6. All these works are based on the idea of constructing specific matrices having special properties by fixing the entries of the SMs in an element-by-element manner at some desirable values. Such an elementwise approach is not permitted in the case of CSMs. A completely different approach based on columnwise manipulation of matrices has been presented in this paper. When limited to the case of SMs, however, the proofs of this paper can be regarded as alternative proofs to the theorems cited above.

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